Fixed Points On Two Classes Of Finite Groups

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Abstract. The ***NG***-groups are groups which consisting of transformations on a non-empty set *A* and have no bijection as its elements. In this paper, we consider the ***NG***-groups by using fixed points.

Keywords: Finite groups, Symmetric groups, Fixed points.

INTRODUCTION

We consider the *NG*-group which consisting of transformations on a non-empty set *A* and the group has no bijection as its element. Recall a permutation group on *A* is a group consisting of bijections from *A* to *A* with respect to compositions of mappings [2],[3]. It is well known that any permutation group on a set *A* with carnality$n$ has order not greater than n![4],[5]. There are some authors, [6], [7], Problem 1.4 in [1], considering groups which consists of non-bijective transformations on *A* where the binary operation is the composition of mappings. Our result is on the orders of such groups are mapping on a non-empty set *A*with respect to function compositions which are not subsets of symmetric groups by using the fixed points.

PROPOSITION 1.1. The maximal of maximum of ***NG*** has $n\geq 3$ elements is $(n-1)\times n$.

i.e. $NG_{Max} = (n-1)\times n$.

THEOREM 1.2. Suppose ***NG*** a group which consists of non-bijective transformations on a non-empty set *A*. For any fixed point $i\in A$,

$|NG_i| \times |i^{NG}| = |NG|$.

REMARK 1.1. The Theorem 1.1 is work for only if $a\in A$ is a fixed point

NG-Groups

In his section, we consider some examples of ***NG***-groups which consisting of transformations on a non-empty set *A***.**

Definition 2.1.The ***NG***-groups are groups which consisting of transformations on a non-empty set *A* and the groups have no bijection as its elements.

Definition 2.2. Let *f* be a transformation mapping on *A*. An elements *a* is a ***fixed*** pointif $f(a) = a$, otherwise it's a ***moved*** point. And, for $g \in$ ***NG*** ,*fix(g)* is the ***number of fixed points***

$a \in A$ such that$ag = a$

i.e. *fix(a)* = the number of fixed points of $a \in A$.

Example 2.1. Consider the set $A = \{1, 2, 3\}$, there are 27 transformations mappings from *A* to*A*. We know the symmetric groups of *A* is$\boldsymbol{S_3}$;
$\boldsymbol{S_3} = \{(1, 2, 3),(2, 3, 1),(3, 1, 2),(1, 3, 2),(3, 2, 1),(2, 1, 3)\}$.
So,$\boldsymbol{S_3}$ is a group. But, it is not an abelian group.

Moreover, there exists some groups that are subsets of *Trans*(*a*), but not subsets of the *Sym(a)***.** We know, every group must contain the identity element. From this point, we will choose an element of *Trans*(*a*) to be the identity element. We repeat this method for all elements of *Trans*(*a*). In addition, we will find all groups that cannot be subset of the symmetric group $\boldsymbol{S_3}$. For all elements of *Trans*(*a*), we introduce the elements of *Trans*(*a*) as:

{(1, 1, 1),(1, 1, 2),(1, 1, 3),(1, 2, 1),(1, 2, 2),(1, 2, 3),(1, 3, 1),(1, 3, 2),(1, 3, 3),
(2, 1, 1),(2, 1, 2),(2, 1, 3),(2, 2, 1),(2, 2, 2),(2, 2, 3),(2, 3, 1),(2, 3, 2),(2, 3, 3),
(3, 1, 1),(3, 1, 2),(3, 1, 3),(3, 2, 1),(3, 2, 2),(3, 2, 3),(3, 3, 1),(3, 3, 2),(3, 3, 3)}.

The groups of order 2 are:

$\boldsymbol{NG_1} = \{(1, 1, 3),(3, 3, 1)\}$, $\boldsymbol{NG_2} = \{(1, 2, 1),(2, 1, 2)\}$,
$\boldsymbol{NG_3} = \{(1, 2, 2),(2, 1, 1)\}$, $\boldsymbol{NG_4} = \{(1, 2, 3),(1, 3, 2)\}$, $\boldsymbol{NG_5} = \{(1, 2, 3),(2, 1, 3)\}$,
$\boldsymbol{NG_6} = \{(1, 2, 3),(3, 2, 1)\}$, $\boldsymbol{NG_7} = \{(1, 3, 3),(3, 1, 1)\}$, $\boldsymbol{NG_8} = \{(2, 2, 3),(3, 3, 2)\}$,
and$\boldsymbol{NG_9} = \{(2, 3, 2),(3, 2, 3)\}$. But, $\boldsymbol{NG_4} = \{(1, 2, 3),(1, 3, 2)\}$, $\boldsymbol{NG_5} = \{(1, 2, 3),(2,1, 3)\}$, and $\boldsymbol{NG_6} = \{(1, 2, 3),(3, 2, 1)\}$ *are subsets of Sym(A).*

The only groups of mapping on a set *A* with respect to function compositions which are not subsets of symmetric groups are: $\boldsymbol{NG_1} = \{(1, 1, 3),(3, 3, 1)\}$, $\boldsymbol{NG_2} = \{(1, 2, 1),(2, 1, 2)\}$, $\boldsymbol{NG_3} = \{(1, 2, 2),(2, 1, 1)\}$, $\boldsymbol{NG_7} = \{(1, 3, 3),(3, 1, 1)\}$, $\boldsymbol{NG_8} = \{(2, 2, 3),(3, 3, 2)\}$, and $\boldsymbol{NG_9} = \{(2, 3, 2),(3, 2, 3)\}$. And, the groups of order 3 are:
{(1, 2, 3),(2, 3, 1),(3, 1, 2)} which is subset of*Sym*(*A*).

Moreover, there are no groups of order greater than order 3 that cannot be subset of *Sym*(A).

So, the only groups of mapping on a set A with respect to function compositions that cannot be subsets of symmetric groups are six groups of order 2;

$\boldsymbol{NG_1} = \{(1, 1, 3),(3, 3, 1)\}$, $\boldsymbol{NG_2} = \{(1, 2, 1),(2, 1, 2)\}$, $\boldsymbol{NG_3} = \{(1, 2, 2),(2, 1, 1)\}$,
$\boldsymbol{NG_7} = \{(1, 3, 3),(3, 1, 1)\}$, $\boldsymbol{NG_8} = \{(2, 2, 3),(3, 3, 2)\}$, $\boldsymbol{NG_9} = \{(2, 3, 2),(3, 2, 3)\}$.

We can see, if $A = \{a, b, c\}$, $\boldsymbol{G} = \{e, f\} \subset A^A$.
Where $e(a) = a$, $e(b) = a$, $e(c) = c$,$f(a) = c$, $f(b) = c$, $f(c) = a$, we get:

$$\boldsymbol{NG}_1 = \{e(a, b, c) = (a, a, c), f(a, b, c) = (c, c, a)\};$$

Where $e(a) = a$, $e(b) = c$, $e(c) = c$, $f(a) = c$, $f(b) = a$, $f(c) = a$, we get:

$$\boldsymbol{NG}_7 = \{e(a, b, c) = (a, c, c), f(a, b, c) = (c, a, a)\};$$

Where $e(a) = a$, $e(b) = b$, $e(c) = b$, $f(a) = b$, $f(b) = b$, $f(c) = a$, we get:

$$\boldsymbol{NG}_3=\{e(a, b, c) = (a, b, b), f(a, b, c) = (b, a, a)\};$$

Where $e(a) = b$, $e(b) = b$, $e(c) = c$, $f(a) = c$, $f(b) = c$, $f(c) = b$, we get:

$$\boldsymbol{NG}_8 = \{e(a, b, c) = (b, b, c), g(a, b, c) = (c, c, b)\};$$

Where $e(a) = a$, $e(b) = b$, $e(c) = a$, $f(a) = b$, $f(b) = a$, $f(c) = b$, we get:

$$\boldsymbol{NG}_2 = \{e(a, b, c) = (a, b, a), f(a, b, c) = (b, a, b)\};$$

Where $e(a) = b$, $e(b) = c$, $e(c) = b$, $f(a) = c$, $f(b) = b$, $f(c) = c$,we get:

$$\boldsymbol{NG}_9 = \{e(a, b, c) = (b, c, b), f(a, b, c) = (c, b, c)\}.$$

So, if A = {a, b, c}, G = {e, f} ⊂ AA, we see e2 = e, ef = fe = f, f 2 = e.

Example 2.2. Suppose $A = \{1, 2, 3\}$. Consider the transformation mapping on A.

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}$$

It has two fixed points {1, 3} since $f(1) = 1$ and $f(3) = 3$. And $2 \in \boldsymbol{A}$ is a movement point because $f(2) \neq 2$ .The number of fixed points in ***NG*** is $|\ fix(f)\ |= 2$ and the number of movement points is $|\ Move(f)\ |= 1$.
i.e. we have two fixed points and one movement point and one movement point.

Suppose m is a movement point in ***NG*** = $\{e, f\}$ where$e = (1, 2, m)$.
If $m = 1$, we get $\boldsymbol{NG_2}$and if $m = 2$, we get $\boldsymbol{NG_3}$,. Moreover, f not has a fixed point. Also, the point 3 is not a loop.
Where $e = (1, m, 3)$, then m can be 1 or 3.
If $m = 1$, then we get $\boldsymbol{NG_1}$ and if $m = 3$, then we get $\boldsymbol{NG_7}$. An addition, f does not have fixed points and 2 not loop.
If $e = (m, 2, 3)$, then$m$ can be 2 or 3.
If $m = 2$, we get $\boldsymbol{NG_8}$ .If $m = 3$, then we get $\boldsymbol{NG_6}$. Moreover, $f$not have fixed points and 1 not a loop.
Now, we can generalization our result for a finite set A. For a finite set A, the identitycould be $e = (1, 2, ..., n - 1, m)$. So, we have $n - 1$ possibility of m, *where* $m = 1, 2, 3, ...,$ or$n - 1$. So, we can get the numbers of maximum groups of ***NG***, we denoted by $\boldsymbol{NG_{Max}}$, of ***G*** that are mapping on aset A with respect to function compositions which are not subsets of symmetric groups by using the fixed points.
The number of maximum of ***NG***which has $n \geq 3$ elements is:

$$\boldsymbol{NG_{\mathbf{Max}}} = (n - 1) \times \left(\frac{n!}{(n-1)! \times 1!} \right)$$

it means

$$\boldsymbol{NG_{Max}} = (n - 1) \times n$$

We can introduce the easy proposition **Proposition 2.1**. The maximal of maximum of ***NG***which has n elements is

$$(n-1)\times n, \text{ ,where } n \geq 3\ .$$
$$\text{i.e}\boldsymbol{NG}_{\text{Max}} = (n-1)\times n.$$

Proof.It easy to prove it by mathematical induction.

We will rewrite the definition of stabilizer of a point as an algebraic concept: it is the set of group elements which fix the point.

Definition 2.3. Let ***NG***be a group which consists of non-bijective transformations on a non-empty set A. Suppose a be an element of A. Then

$$\boldsymbol{Satb}(a) = \boldsymbol{NG}_a = \{g \in \boldsymbol{NG} : g.a = a\}$$

is called the stabilizer of a and consists of all the transformation ***NG*** that produce group fixed points in A.
i.e., that send a to itself.

Let ***NG*** be a group which consists of non-bijective transformations on a non-empty set A. If $a \in A$, then the point-stabilizer $\boldsymbol{NG}_{\text{a}}$ is the subgroup of ***NG*** formed by the elements which*fix(a)*. Moreover, we will write the definition of orbit of a point as a geometric concept: it is the set of places where the point can be moved by the group action and the size of an orbit as its length.

Definition 2.4.Let ***NG*** be a group which consists of non-bijective transformations on a non-empty set A. Suppose a be an element of A. Then, the group ***NG*** containing $a \in A$ is

$$\text{I}^{\boldsymbol{NG}} = Orb(a) = \{ag : g \in \boldsymbol{NG}\}\text{. And, we called it the } \textbf{\textit{orbit}} \text{ of } a.$$

Let ***NG*** be a group which consists of non-bijective transformations on a non-empty set A. If $a \in A$, then the orbit of a is the subgroup of A formed by the elements which *fix(a)*.

Example 2.3. Let $A = \{1, 2, 3\}$ and $\boldsymbol{NG} = \{(1, 1, 3),(3, 3, 1)\}$. So, the fixed points are $\{1, 3\}$. The group ***NG*** containing $1 \in A$ is
$1^{\boldsymbol{NG}} = \{ig : g \in \boldsymbol{NG}\} = \{1, 3\} = 2^{\boldsymbol{NG}} = 3^{\boldsymbol{NG}}$.
The Stabilizer in a group ***NG*** of i is$\boldsymbol{NG}_{\boldsymbol{i}} = \{g \in \boldsymbol{NG} : ig = i\}$.

$$\boldsymbol{NG}_1 = \{(1, 1, 3)\}.$$

Note that, $\quad |\ \boldsymbol{NG}_1\ | \times |\ 1^{\mathbf{NG}}\ | = 1 \times 2 = 2 = |\ \boldsymbol{NG}\ |$.

Remark 2.1. In previous example, if we take $i = 2$, then $\boldsymbol{NG}_2 = \varphi$.
So, $| \boldsymbol{NG}_2 | \times | 2^{\boldsymbol{NG}} | = 0 \times 2 \neq | \boldsymbol{NG} |$.

Proposition 2.2. Suppose $\boldsymbol{NG}$ a group which consists of non-bijective transformations on a non-empty set A. For any fixed point $i \in A$,

$$| \boldsymbol{NG_i} | \times | i^{\mathbf{NG}} | = | \boldsymbol{NG} |.$$

Proof. Suppose $i^{\mathbf{NG}} = \{i1, ..., ir\}$ and $K = \boldsymbol{NG}^{i} = \{k1, ..., Km\}$.
Choose elements $g1 = 1, g2, ..., gr$ such that $igj = ij$
for each j.
Construct(some of) the group elements like this .

k1g1 k1g2 . . . k1gr
k2g1 k2g2 . . . k2gr
. . . . . .
. . . . . .
. . . . . .
ksg1 ksg2 . . . ksgr
□
The column j has elements form the cosetKgj.
In cosetKgj, every element maps i to ij .
So, this array contains exactly r × s distinct elements of NG.

Now, we claim every element $g \in \boldsymbol{NG}$ as exactly once in array above.
We can find ig. By definition of fixed points and orbit$ig \in i\mathbf{NG}$. So, $ig = ij$ for some j.
Calculate $k := gg-1$. Then K is map from$i$ to i.
So, $k \in K = \boldsymbol{NG}_1$. Hence $g = kgj$ lies in column j.
Counting: $| \boldsymbol{NG} |$ =number of elements in the array

$$= r\times = | i^{\boldsymbol{NG}} | \times | \boldsymbol{NG}_i |$$

where $| i^{\boldsymbol{NG}} |$ as the length of $i^{\boldsymbol{NG}.}$

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